EE 508
Lecture 16

Filter Transformations

Lowpass to Highpass
Lowpass to Band-reject

Filter Synthesis

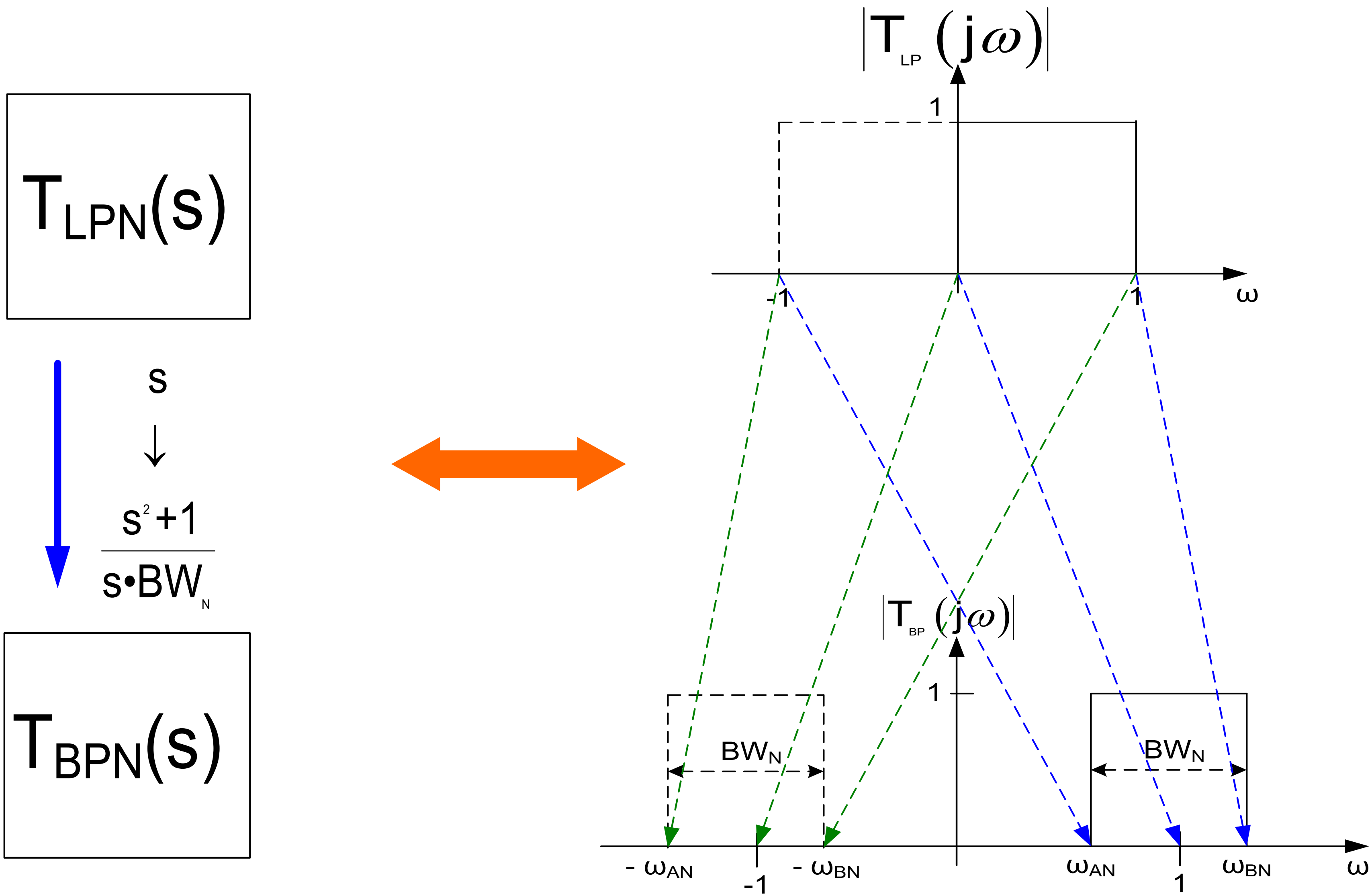
Standard LP to BP Transformation

$$s \rightarrow \frac{s^2+1}{s \bullet BW_N}$$

- Standard LP to BP transform is a variable mapping transform
- Maps jω axis to jω axis
- Maps LP poles to BP poles
- Preserves basic shape but warps frequency axis
- Doubles order
- Pole Q of resultant band-pass functions can be very large for narrow pass-band
- Sequencing of frequency scaling and transformation does not affect final function

Review from Last Time

Standard LP to BP Transformation

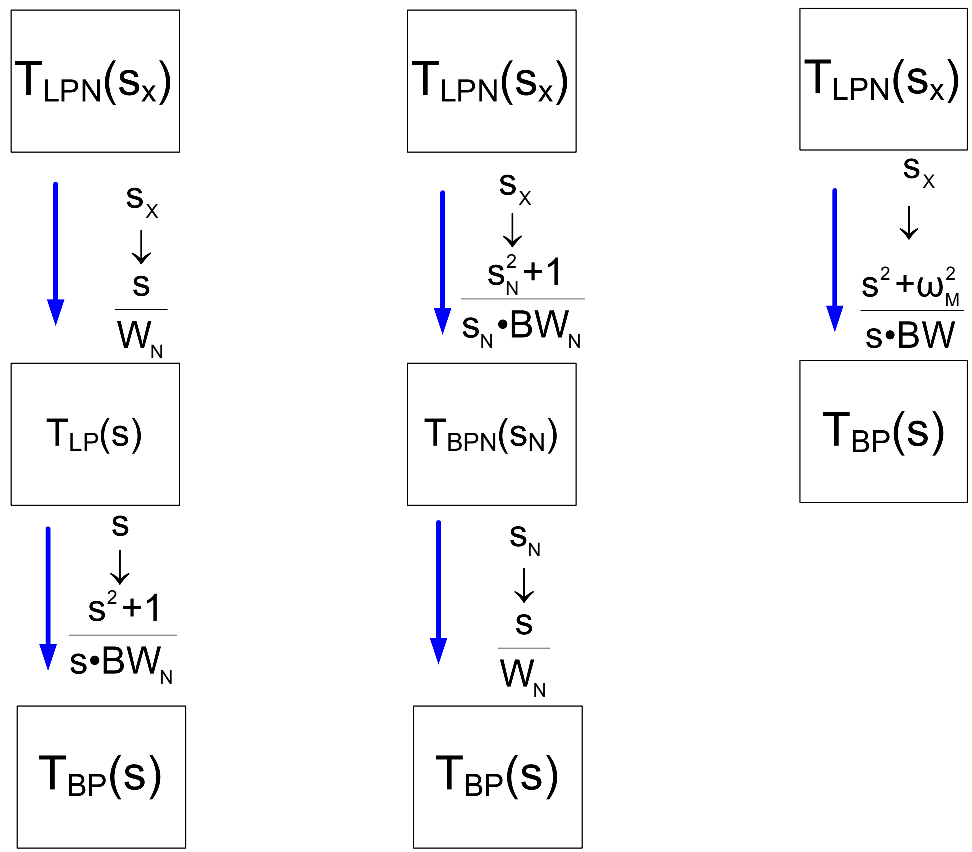


Review from Last Time

Standard LP to BP Transformation

Frequency and s-domain Mappings - Denormalized

(subscript variable in LP approximation for notational convenience)



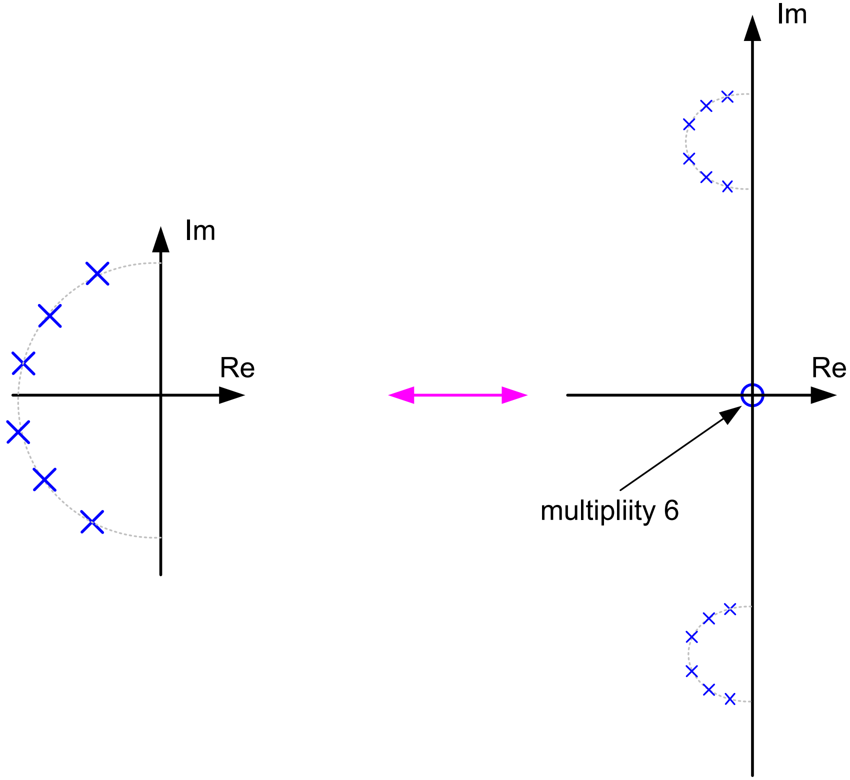
All three approaches give same approximation

Which is most practical to use? Often none of them !

Standard LP to BP Transformation

Pole Mappings

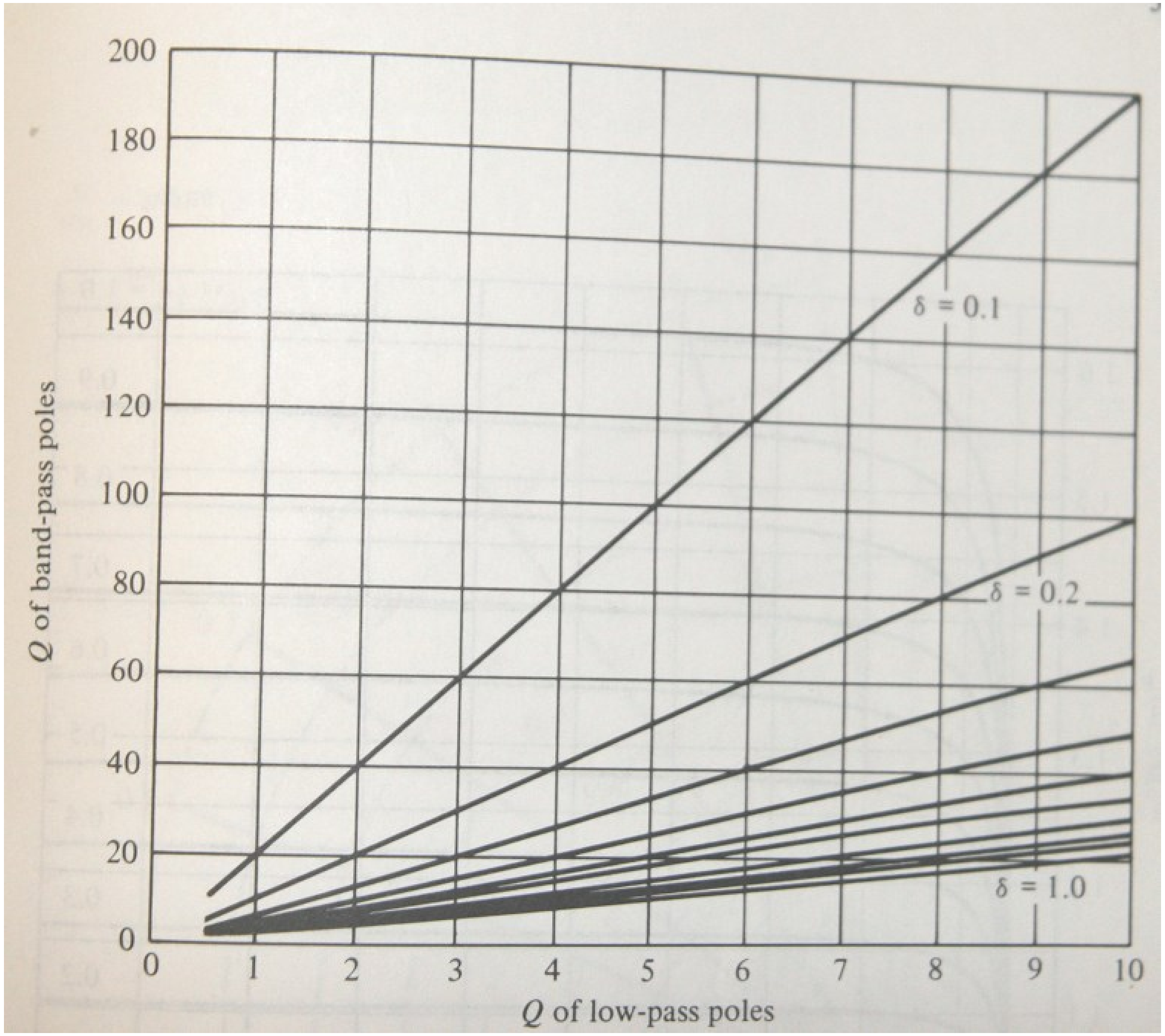
$$p \leftarrow \frac{p_X \bullet BW_N \pm \sqrt{\left(BW_N \bullet p_X\right)^2 - 4}}{2}$$



Note doubling of poles, addition of zeros, and likely Q enhancement

LP to BP Transformation

Pole Q of BP Approximations



$$\delta = \left(\frac{\mathrm{BW}}{\omega_{\mathrm{M}}}\right)\omega_{0\mathrm{LP}}$$

$$Q_{\mathrm{BPL}} = Q_{\mathrm{BPH}} = \frac{Q_{\mathrm{LP}}}{\sqrt{2}}\sqrt{1+\frac{4}{\delta^2}+\sqrt{\left(1+\frac{4}{\delta^2}\right)^2-\frac{4}{\delta^2 Q_{2\mathrm{LP}}^2}}}$$

LP to BP Transformation

Pole locations vs Q_{LP} and δ

$$\delta = \left(\frac{BW}{\omega_M}\right)\omega_{0LP}$$

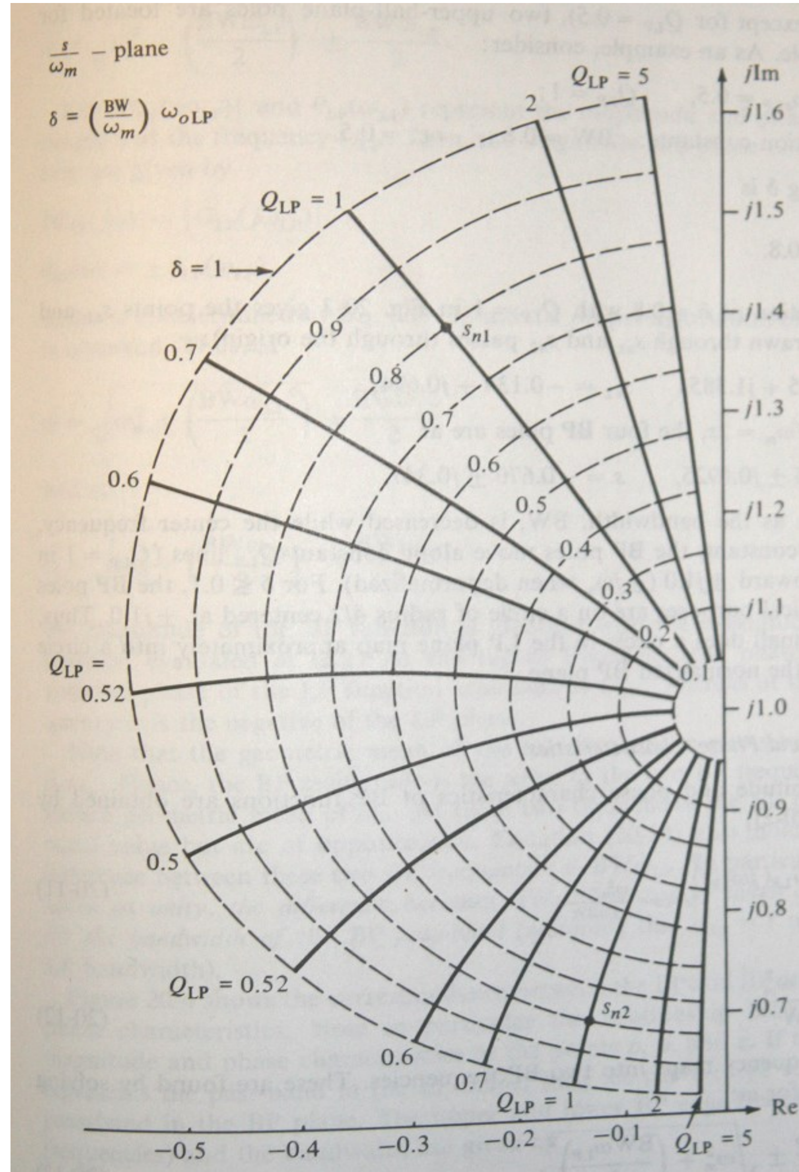
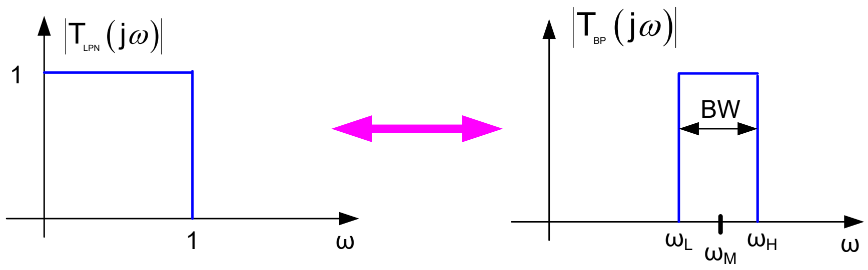


Fig. 20-3 Upper-half-plane band-pass poles as a function of Q_{LP} and δ

LP to BP Transformation



Classical BP Approximations

- Butterworth
- Chebyschev
- Elliptic
- Bessel

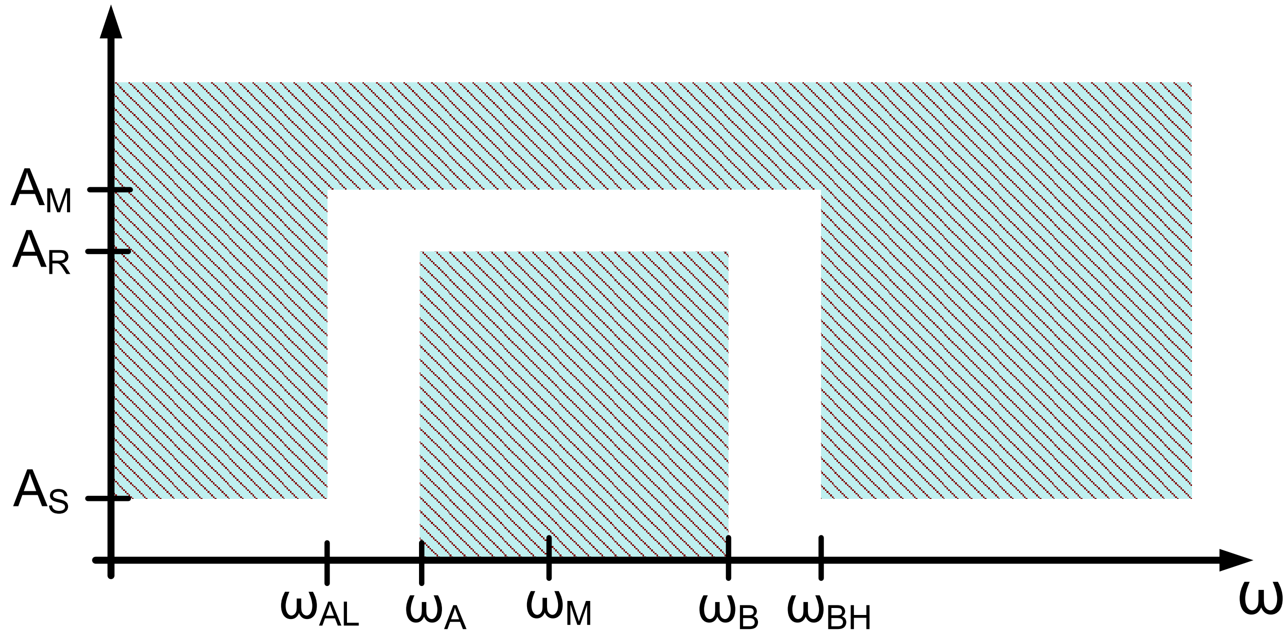
Obtained by the LP to BP transformation of the corresponding LP approximations

Standard LP to BP Transformation

$$s \rightarrow \frac{s^2+1}{s \bullet BW_N}$$

- Standard LP to BP transform is a variable mapping transform
- Maps jω axis to jω axis
- Maps LP poles to BP poles
- Maps LP zeros to BP zeros
- Preserves basic shape but warps frequency axis
- Doubles order
- Introduces additional zeros at origin (number equal half the order)
- Pole Q of resultant band-pass functions can be very large for narrow pass-band
- Sequencing of frequency scaling and transformation does not affect final function

Example 1: Obtain an approximation that meets the following specifications



$$BW=\omega_B-\omega_A$$

$$\omega_M=\sqrt{\omega_B \bullet \omega_A}$$

Assume that ω_{AL}, ω_{BH} and ω_M satsify

$$\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL} \bullet BW}=\frac{\omega_{BH}^2-\omega_M^2}{\omega_{BH} \bullet BW}$$

Recall from above:

Standard LP to BP Transformation

Frequency and s-domain Mappings - Denormalized

(subscript variable in LP approximation for notational convenience)

$T_{LPN}(s_X)$

$$s_X \to \frac{s^2+\omega_M^2}{s\bullet BW}$$

$T_{BP}(s)$

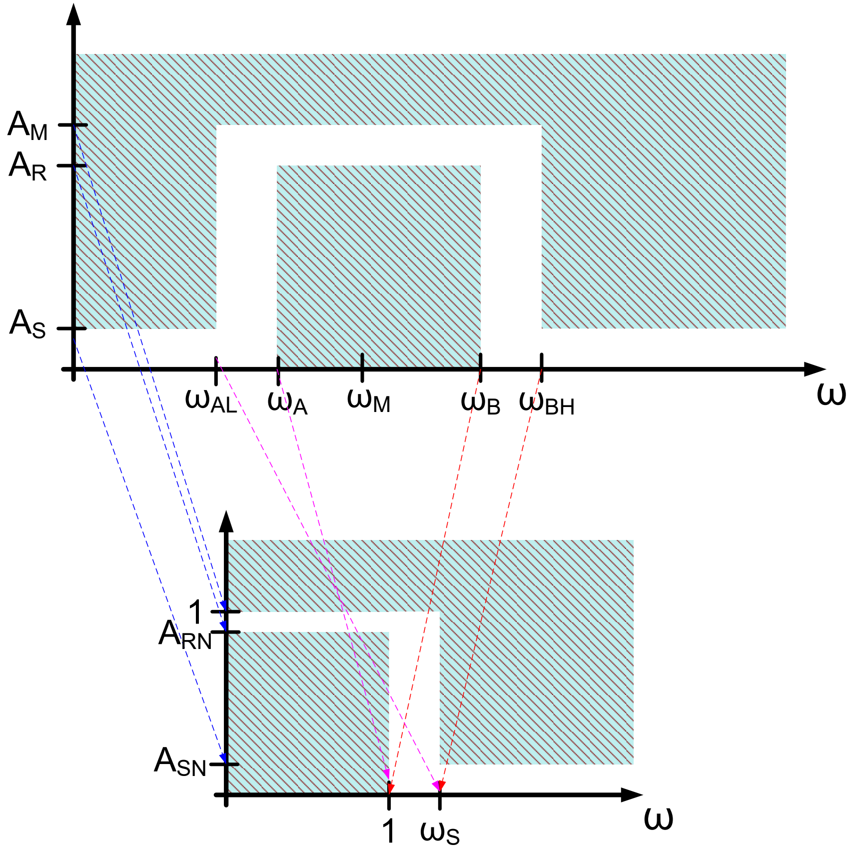
$$s_X \to \frac{s^2+\omega_M^2}{s\bullet BW}$$

$$\omega_X \to \frac{\omega^2-\omega_M^2}{\omega\bullet BW}$$

$$s \leftarrow \frac{s_X\bullet BW \pm \sqrt{\left(BW\bullet s_X\right)^2-4\omega_M^2}}{2}$$

$$\omega \leftarrow \frac{\omega_X\bullet BW \pm \sqrt{\left(BW\bullet \omega_X\right)^2+4\omega_M^2}}{2}$$

Example 1: Obtain an approximation that meets the following specifications



$$A_{RN} = \frac{A_R}{A_M}$$

$$A_{SN} = \frac{A_S}{A_M}$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{A_R}{A_M}$$

$$\varepsilon = \sqrt{\left(\frac{A_M}{A_R}\right)^2 - 1}$$

$$\omega_S = \frac{\omega_M^2 - \omega_{AL}^2}{\omega_{AL} \bullet BW}$$

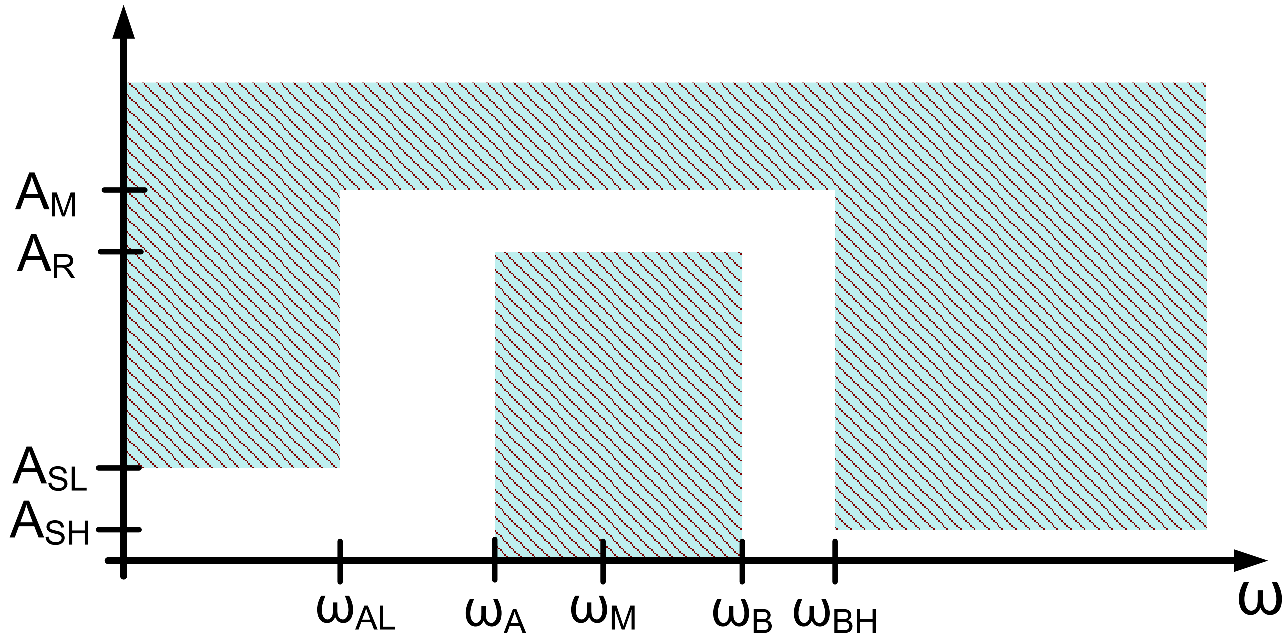
$$BW = \omega_B - \omega_A$$

$$\omega_M = \sqrt{\omega_B \bullet \omega_A}$$

$$\frac{\omega_M^2 - \omega_{AL}^2}{\omega_{AL} \bullet BW} = \frac{\omega_{BH}^2 - \omega_M^2}{\omega_{BH} \bullet BW}$$

(actually ω_A and ω_{AL} that map to -1 and $-\omega_S$ respectively but show 1 and ω_S for convenience)

Example 2: Obtain an approximation that meets the following specifications



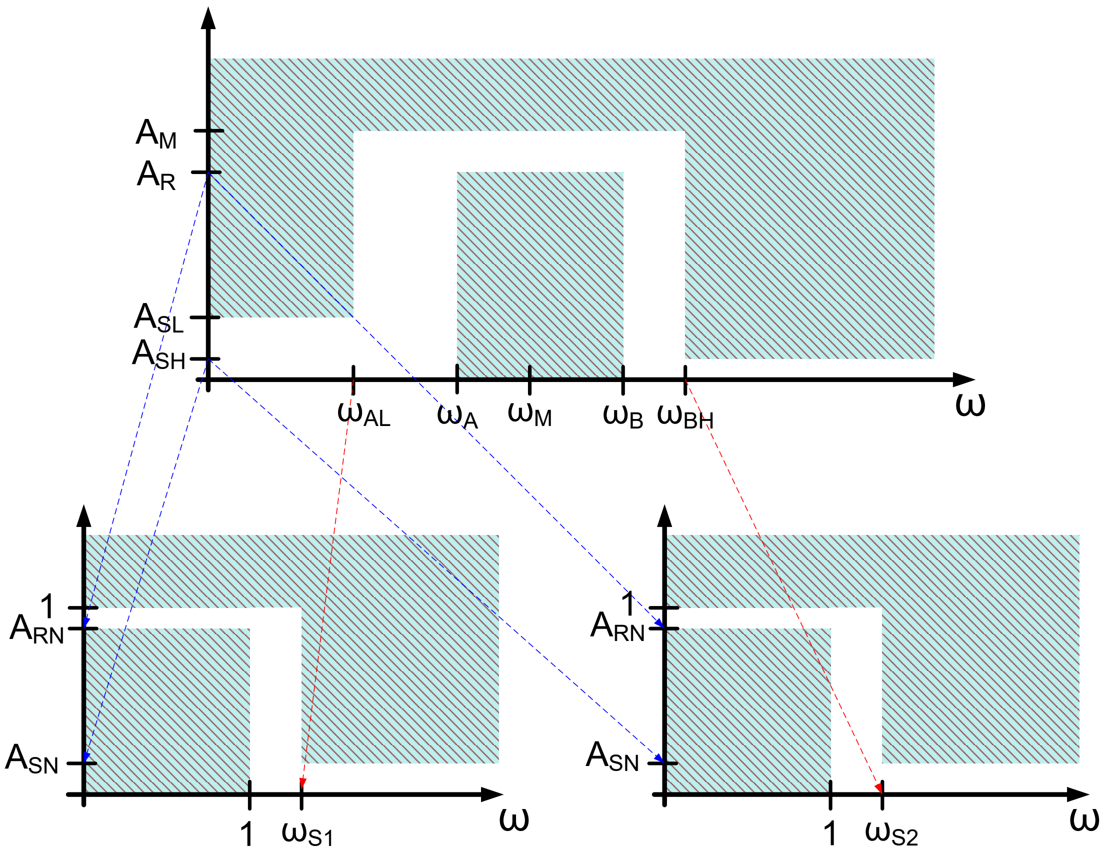
$$BW=\omega_B-\omega_A$$

$$\omega_M=\sqrt{\omega_B \bullet \omega_A}$$

In this example,

$$\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL}\bullet BW}\neq\frac{\omega_{BH}^2-\omega_M^2}{\omega_{BH}\bullet BW}$$

Example 2: Obtain an approximation that meets the following specifications



$$A_{RN}=\frac{A_R}{A_M}$$

$$\frac{1}{\sqrt{1+\varepsilon^2}}=\frac{A_R}{A_M}$$

$$A_{SN}=\min\left\{\frac{A_{SH}}{A_M},\frac{A_{SL}}{A_M}\right\}$$

$$\varepsilon=\sqrt{\left(\frac{A_M}{A_R}\right)^2-1}$$

$$\omega_{S1}=\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL}\bullet BW}$$

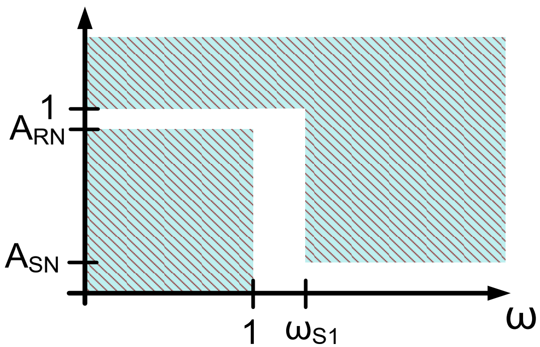
$$\omega_{S2}=\frac{\omega_{BH}^2-\omega_M^2}{\omega_{BH}\bullet BW}$$

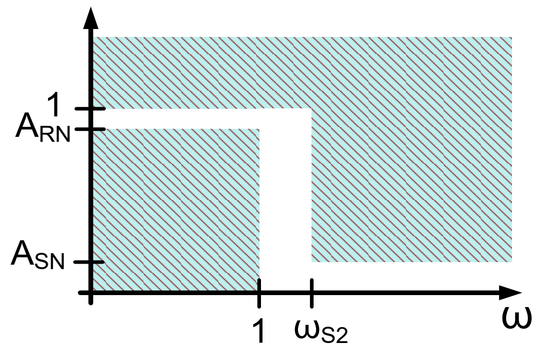
$$BW=\omega_B-\omega_A$$

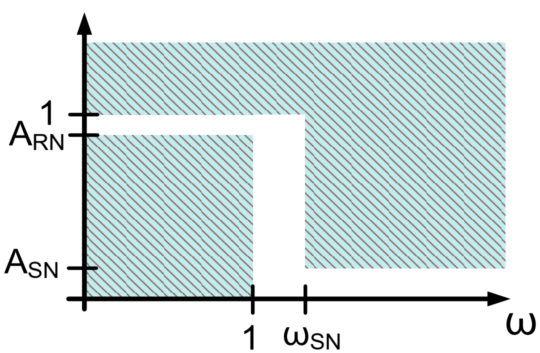
$$\omega_M=\sqrt{\omega_B\bullet\omega_A}$$

$$\omega_{SN}=\min\{\omega_{S1},\omega_{S2}\}$$

Example 2: Obtain an approximation that meets the following specifications







$$A_{RN}=\frac{A_R}{A_M}$$

$$\frac{1}{\sqrt{1+\varepsilon^2}}=\frac{A_R}{A_M}$$

$$A_{SN}=\min\left\{\frac{A_{SH}}{A_M},\frac{A_{SL}}{A_M}\right\}$$

$$\varepsilon=\sqrt{\left(\frac{A_M}{A_R}\right)^2-1}$$

$$\omega_{SN}=\min\{\omega_{S1},\omega_{S2}\}$$

$$\omega_{S1}=\frac{\omega_M^2-\omega_{AL}^2}{\omega_{AL}\bullet BW}$$

$$\omega_{S2}=\frac{\omega_{BH}^2-\omega_M^2}{\omega_{BH}\bullet BW}$$

$$BW=\omega_B-\omega_A$$

$$\omega_M=\sqrt{\omega_B\bullet\omega_A}$$

$$\omega_{SN}=\min\{\omega_{S1},\omega_{S2}\}$$

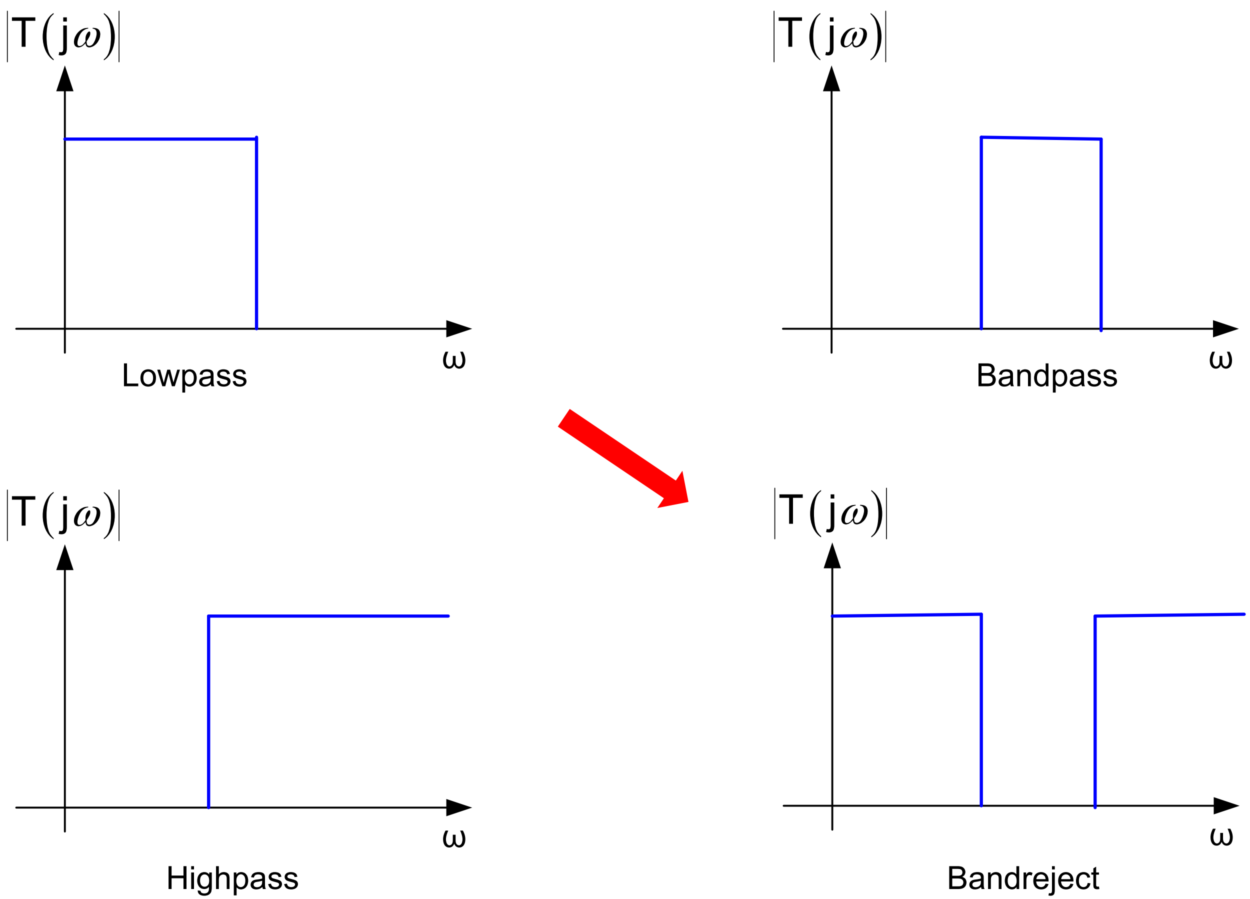
Filter Transformations

→ Lowpass to Bandpass **(LP to BP)**
Lowpass to Highpass **(LP to HP)**
Lowpass to Band-reject **(LP to BR)**

- Approach will be to take advantage of the results obtained for the standard LP approximations

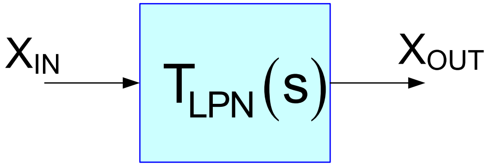
- Will focus on flat passband and zero-gain stop-band transformations

Flat Passband/Stopband Filters

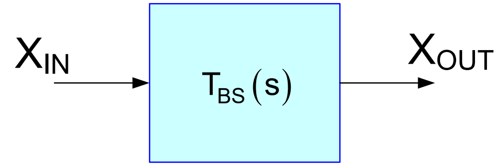


LP to BS Transformation

Strategy: As was done for the LP to BP approximations, will use a variable mapping strategy that maps the imaginary axis in the s-plane to the imaginary axis in the s-plane so the basic shape is preserved.



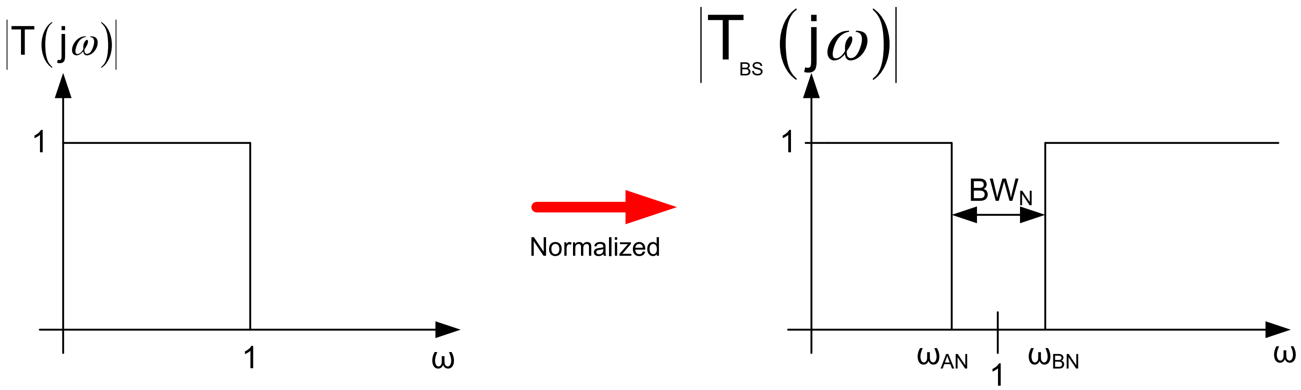
$$s \rightarrow f(s)$$



$$T_{BS}(s) = T_{LPN}(f(s))$$

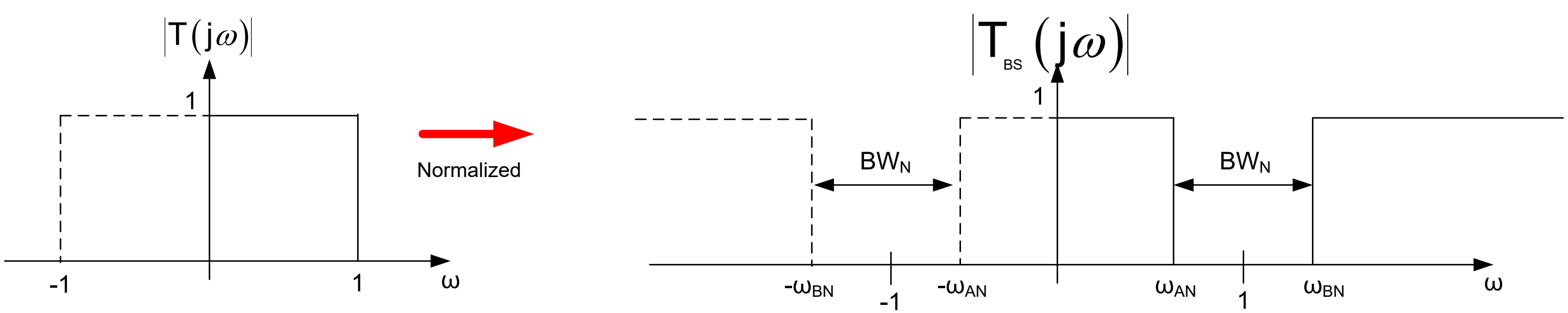
$$f(s) = \frac{\sum_{i=0}^{m_T} a_{Ti} s^i}{\sum_{i=0}^{n_T} b_{Ti} s^i}$$

LP to BS Transformation



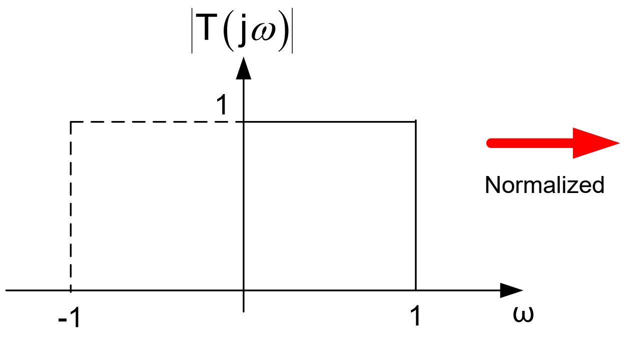
$$BW_N = \omega_{BN} - \omega_{AN}$$

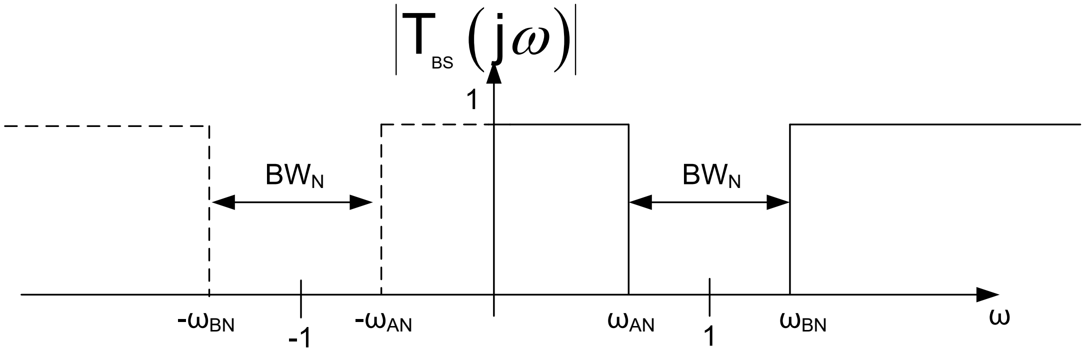
$$\sqrt{\omega_{AN}\omega_{BN}} = 1$$



Standard LP to BS Transformation

Mapping Strategy:





Variable Mapping Strategy to Preserve Shape of LP function:

$F_N(s)$ should

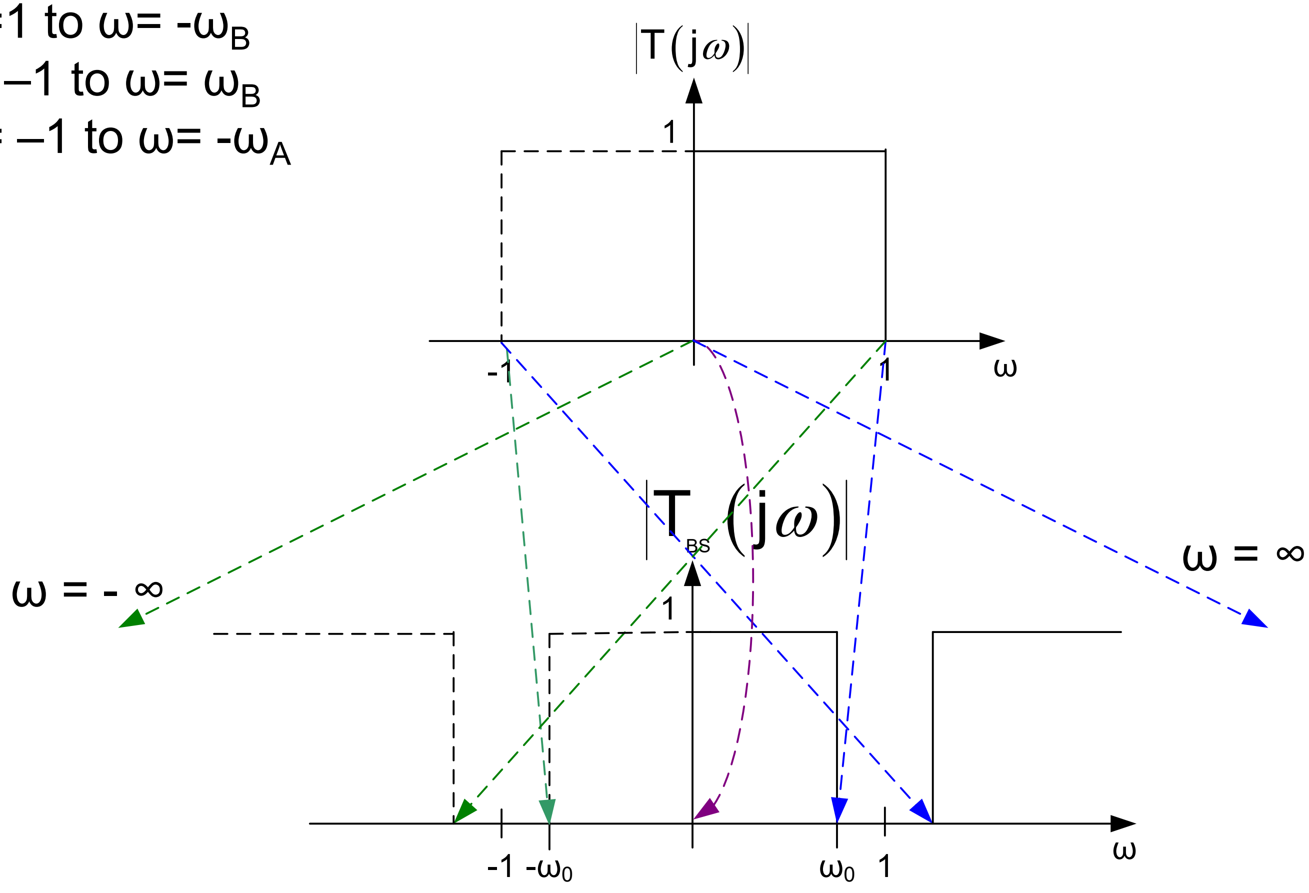
map s=0 to s=± j∞
map s=0 to s= j0
map s=j1 to s=jω_A
map s=j1 to s=-jω_B
map s=-j1 to s=jω_B
map s=-j1 to s=-jω_A

→

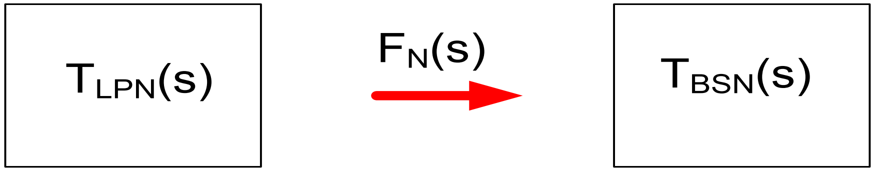
map ω=0 to ω = ±∞
map ω=0 to ω = 0
map ω=1 to ω = ω_A
map ω=1 to ω= -ω_B
map ω= –1 to ω= ω_B
map ω= –1 to ω= -ω_A

Standard LP to BS Transformation

map ω=0 to ω = ±∞
map ω=0 to ω = 0
map ω=1 to ω = ω_A
map ω=1 to ω= $-\omega_B$
map ω= –1 to ω= ω_B
map ω= –1 to ω= $-\omega_A$



Standard LP to BS Transformation



Mapping Strategy: consider variable mapping transform

$F_N(s)$ should

map s=0 to s=± j∞
map s=0 to s= j0
map s=j1 to s=jω_A
map s=j1 to s=-jω_B
map s=-j1 to s=jω_B
map s=-j1 to s=-jω_A

→

map ω=0 to ω = ±∞
map ω=0 to ω = 0
map ω=1 to ω = ω_A
map ω=1 to ω= -ω_B
map ω= –1 to ω= ω_B
map ω= –1 to ω= -ω_A

Consider variable mapping

$$T_{LPN}\left(F_N(s)\right) = T_{BSN}(s)\Big|_{s=\frac{s \bullet BW_N}{s^2+1}}$$

$$s \rightarrow \frac{s \bullet BW_N}{s^2+1}$$

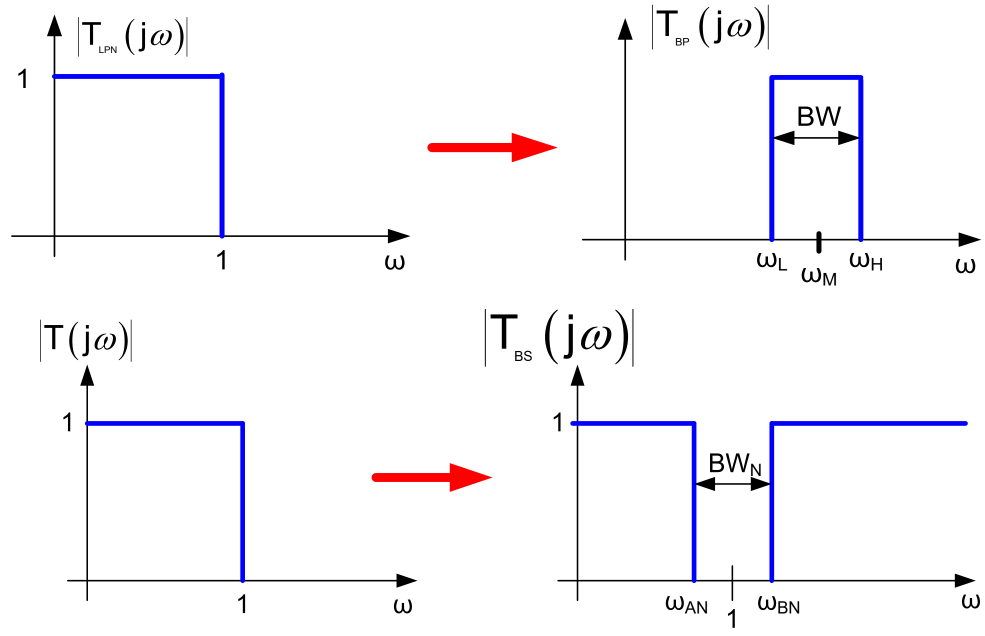
Comparison of Transforms

LP to BP

$$s \rightarrow \frac{s^2+1}{s \bullet BW_N}$$

LP to BS

$$s \rightarrow \frac{s \bullet BW_N}{s^2+1}$$



Standard LP to BS Transformation

Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

$$T_{LPN}(s_X)$$

$$s_X \downarrow \frac{s \bullet BW_N}{s^2+1}$$

$$T_{BSN}(s)$$

$$s_X \rightarrow \frac{s \bullet BW_N}{s^2+1}$$

$$\omega_X \rightarrow \frac{\omega \bullet BW_N}{1\text{-}\omega^2}$$

$$s \leftarrow \frac{1}{2}\frac{BW_N}{s_X} \pm \frac{1}{2}\sqrt{\left(\frac{BW_N}{s_X}\right)^2 - 4}$$

$$\omega \leftarrow \frac{-1}{2}\frac{BW_N}{\omega_X} \pm \frac{1}{2}\sqrt{\left(\frac{BW_N}{\omega_X}\right) + 4}$$

Standard LP to BS Transformation

Un-normalized Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

$T_{LPN}(s_X)$

$$s_X \downarrow \frac{s \bullet BW}{s^2 + \omega_M^2}$$

$T_{BS}(s)$

$$s_X \rightarrow \frac{s \bullet BW}{s^2 + \omega_M^2}$$

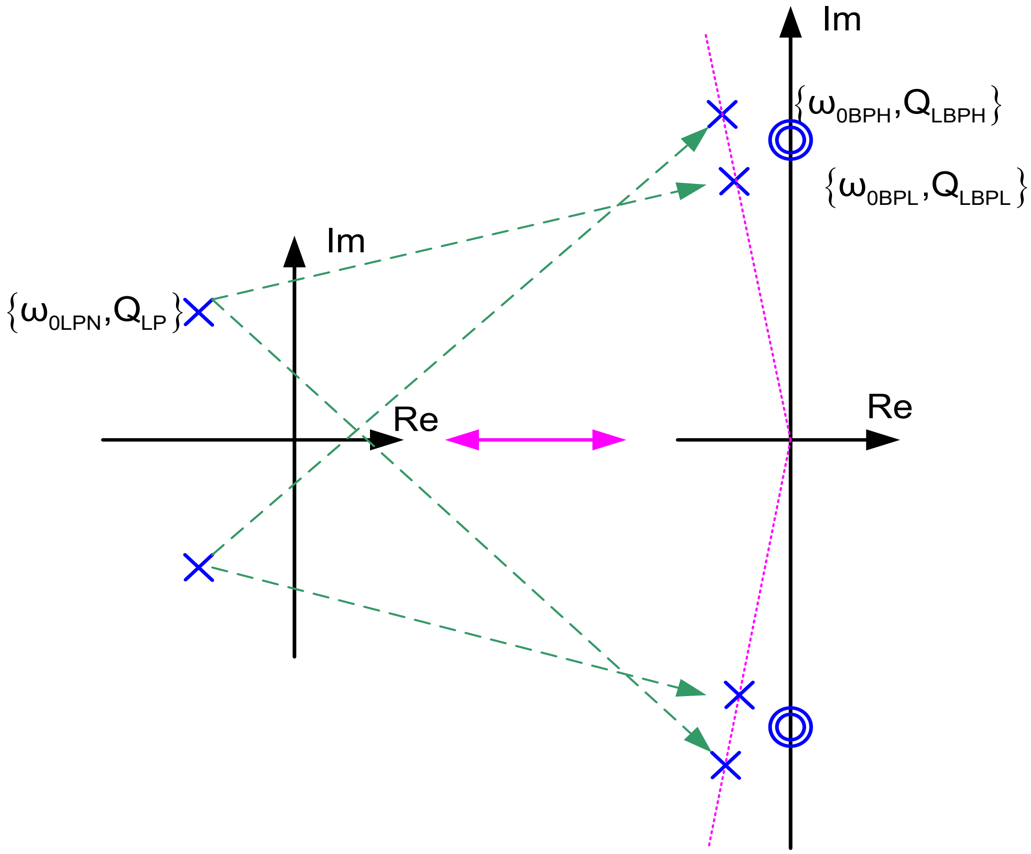
$$\omega_X \rightarrow \frac{\omega \bullet BW}{\omega_M^2 - \omega^2}$$

$$s \leftarrow \frac{1}{2}\frac{BW}{s_X} \pm \frac{1}{2}\sqrt{\left(\frac{BW}{s_X}\right)^2 - 4\omega_M^2}$$

$$\omega \leftarrow \frac{-1}{2}\frac{BW}{\omega_X} \pm \frac{1}{2}\sqrt{\left(\frac{BW}{\omega_X}\right) + 4\omega_M^2}$$

Standard LP to BS Transformation

Pole Mappings

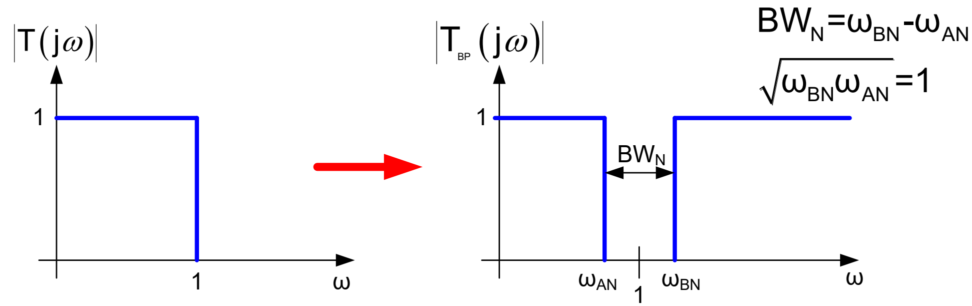


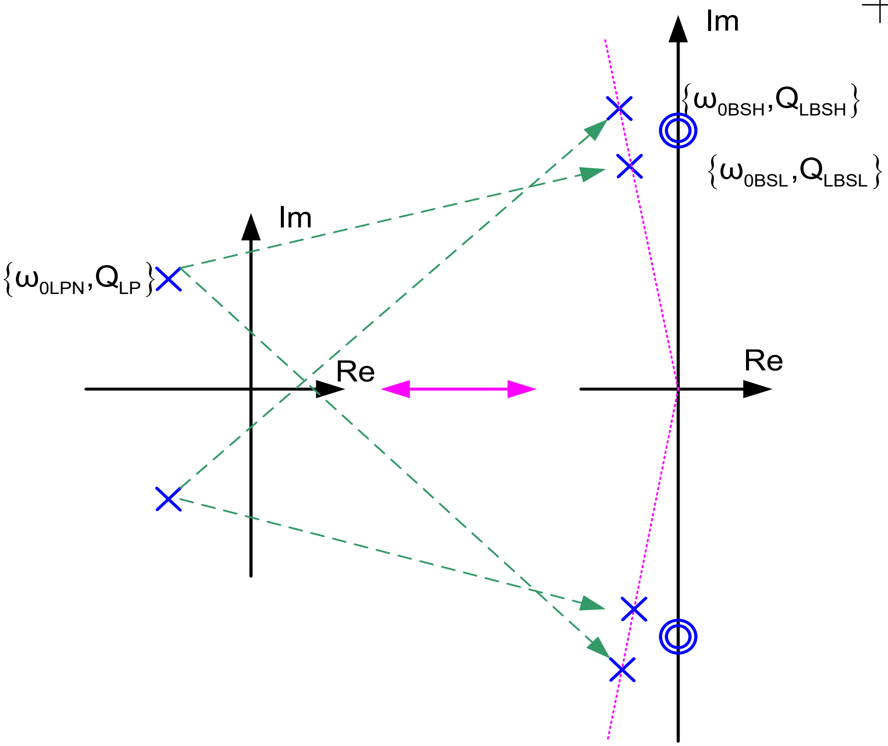
Can show that the upper hp pole maps to one upper hp pole and one lower hp pole as shown. Corresponding mapping of the lower hp pole is also shown

- Poles lie on a constant-Q line
- Zeros at $\pm$ j1 (normalized) or at $\pm j\omega_M$ (un-normalized) of multiplicity n

LP to BS Transformation

Pole Q of BS Approximations





Define: $\gamma = \left(\dfrac{BW}{\omega_M \omega_{0LPN}}\right)$ $\quad BW=\omega_B-\omega_A$ $\quad \sqrt{\omega_B\omega_A}=\omega_M$

It can be shown that

$$Q_{BSL} = Q_{BSH} = \frac{Q_{LP}}{\sqrt{2}}\sqrt{1+\frac{4}{\gamma^2}+\sqrt{\left(1+\frac{4}{\gamma^2}\right)^2-\frac{4}{\gamma^2 Q_{LP}^2}}}$$

For γ small, $\quad Q_{BS} \cong \dfrac{2Q_{LP}}{\gamma}$

It can be shown that

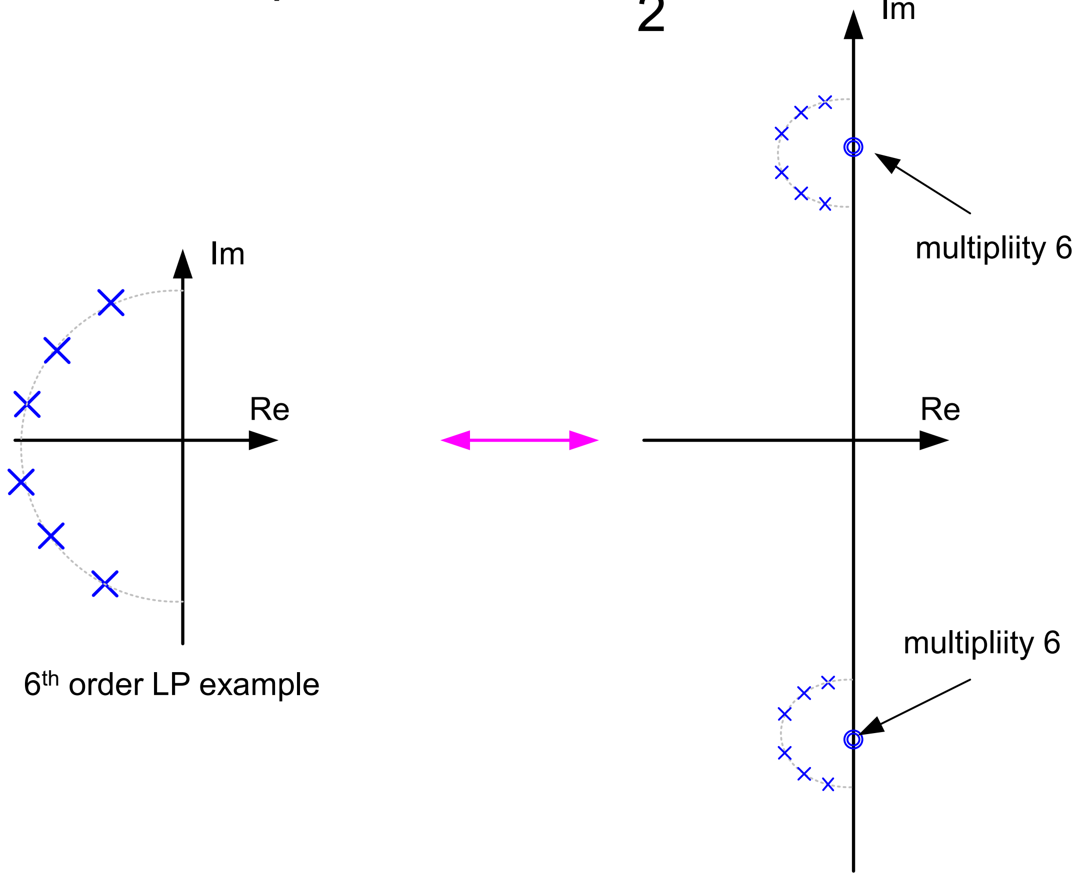
$$\omega_{0BS} = \frac{\omega_M}{2}\left[\gamma\frac{Q_{BS}}{Q_{LP}} \pm \sqrt{\left(\gamma\frac{Q_{BS}}{Q_{LP}}\right)^2-4}\right]$$

Note for γ **small, Q_{BS} can get very large**

Standard LP to BS Transformation

Pole Mappings

$$p \leftarrow \frac{\frac{BW_N}{p_X} \pm \sqrt{\left(\frac{BW_N}{p_X}\right)^2 - 4}}{2}$$



Note doubling of poles, addition of zeros, and likely Q enhancement

Standard LP to BS Transformation

$$s_X \rightarrow \frac{s \bullet BW}{s^2 + \omega_M^2}$$

- Standard LP to BS transformation is a variable mapping transform
- Maps jω axis to jω axis in the s-plane
- Preserves basic shape of an approximation but warps frequency axis
- Order of BS approximation is double that of the LP Approximation
- Pole Q and ω_0 expressions are identical to those of the LP to BP transformation
- Pole Q of BS approximation can get very large for narrow BW
- Other variable transforms exist but the standard is by far the most popular

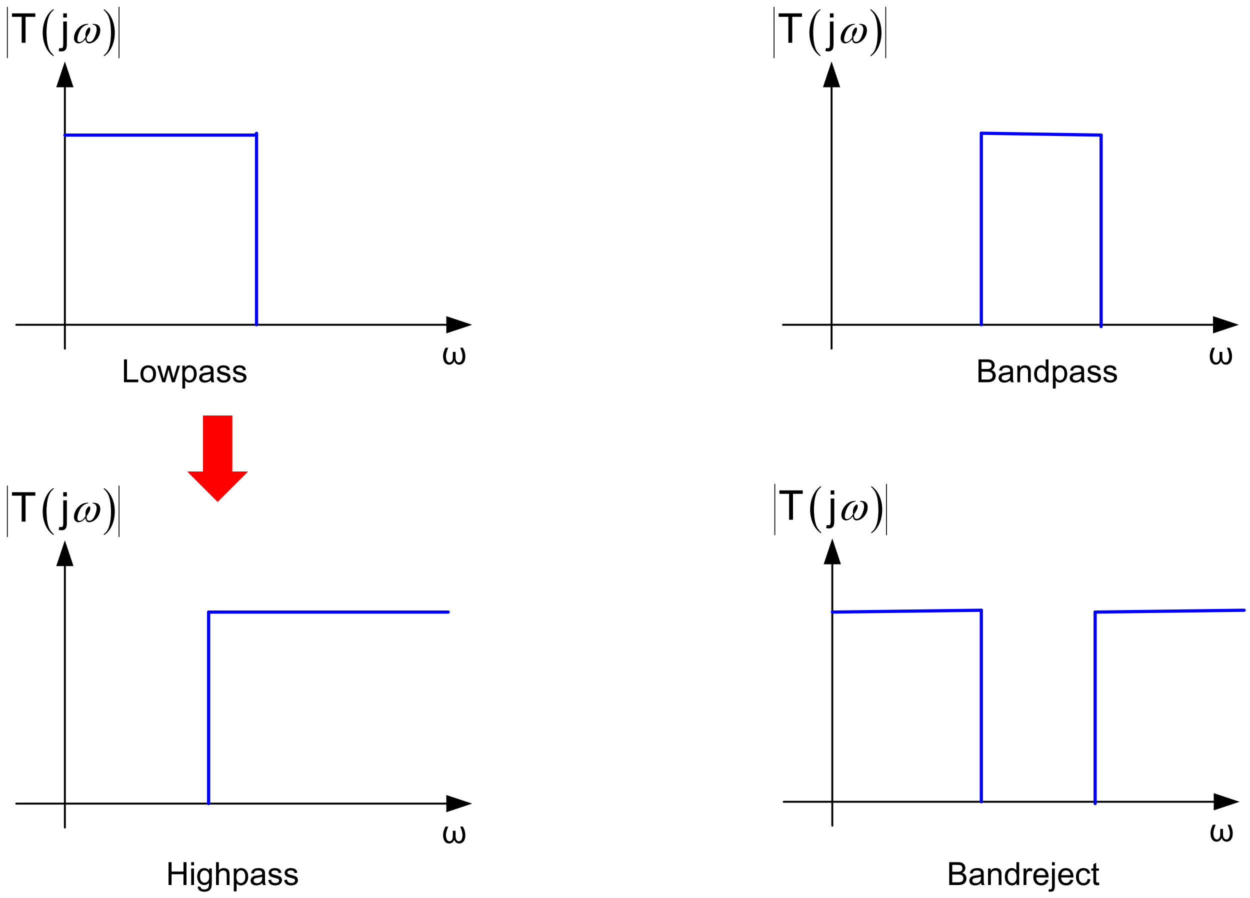
Filter Transformations

Lowpass to Bandpass **(LP to BP)**
→ Lowpass to Highpass **(LP to HP)**
Lowpass to Band-reject **(LP to BR)**

- Approach will also be to take advantage of the results obtained for the standard LP approximations

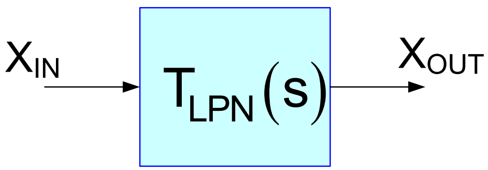
- Will focus on flat passband and zero-gain stop-band transformations

Flat Passband/Stopband Filters

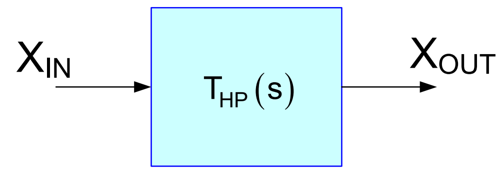


LP to HP Transformation

Strategy: As was done for the LP to BP approximations, will use a variable mapping strategy that maps the imaginary axis in the s-plane to the imaginary axis in the s-plane so the basic shape is preserved.



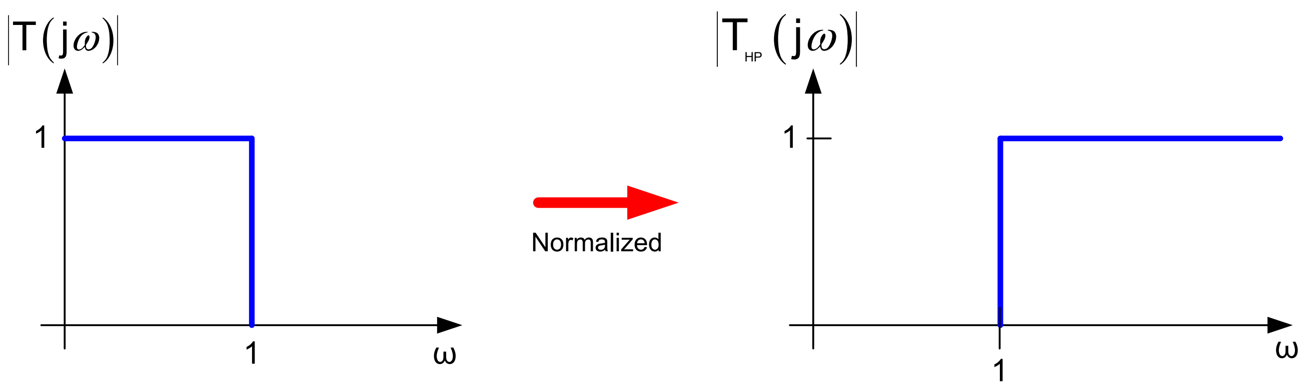
$$s \rightarrow f(s)$$

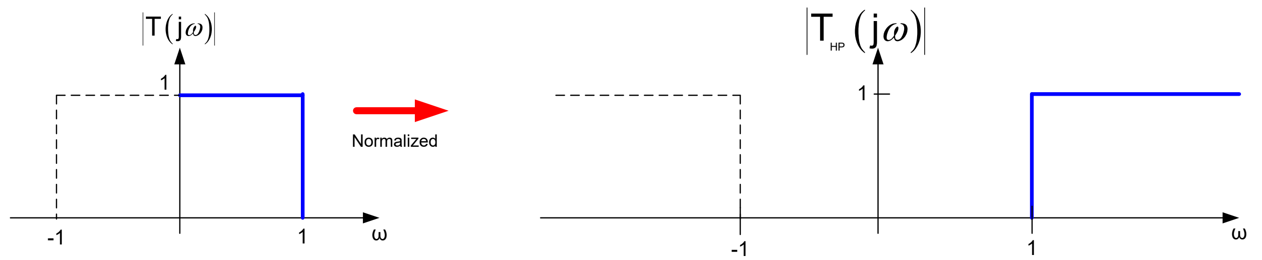


$$T_{HP}(s) = T_{LPN}(f(s))$$

$$f(s) = \frac{\sum_{i=0}^{m_T} a_{Ti} s^i}{\sum_{i=0}^{n_T} b_{Ti} s^i}$$

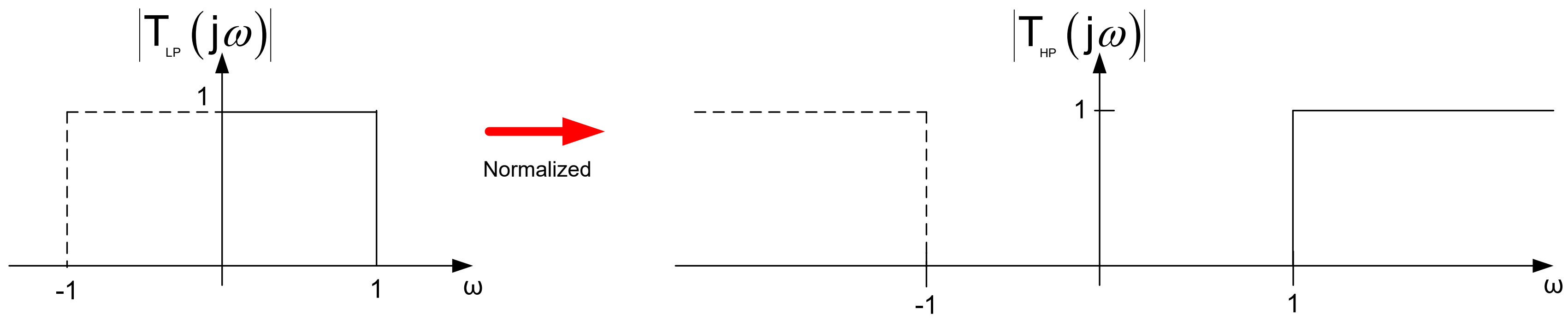
LP to HP Transformation





Standard LP to HP Transformation

Mapping Strategy:



Variable Mapping Strategy to Preserve Shape of LP function:

$F_N(s)$ should

map s=0 to s=± j∞
map s=j1 to s=-j1
map s= –j1 to s=j1

→

map ω=0 to ω=∞
map ω=1 to ω=-1
map ω= –1 to ω=1

Standard LP to HP Transformation

$T_{LPN}(s)$ $\xrightarrow{F_N(s)}$ $T_{HPN}(s)$

Mapping Strategy: consider variable mapping transform

$F_N(s)$ should

map s=0 to s=± j∞
map s=j1 to s=-j1
map s= –j1 to s=j1

→

map ω=0 to ω=∞
map ω=1 to ω=-1
map ω= –1 to ω=1

Consider variable mapping

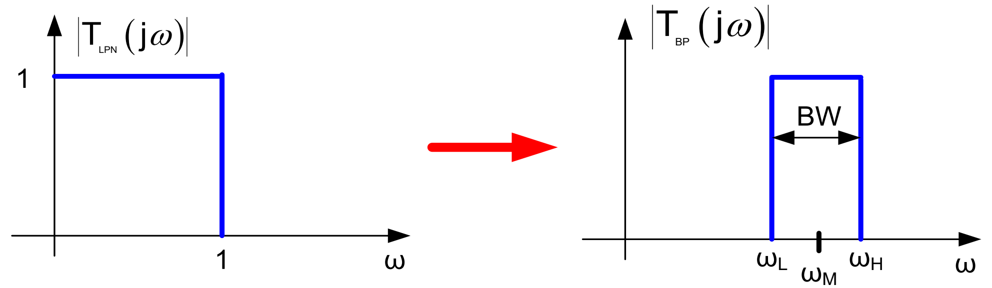
$$T_{LPN}\left(F(s)\right) = T_{LPN}\left(s\right)\Big|_{s=\frac{1}{s}}$$

$$s \rightarrow \frac{1}{s}$$

Comparison of Transforms

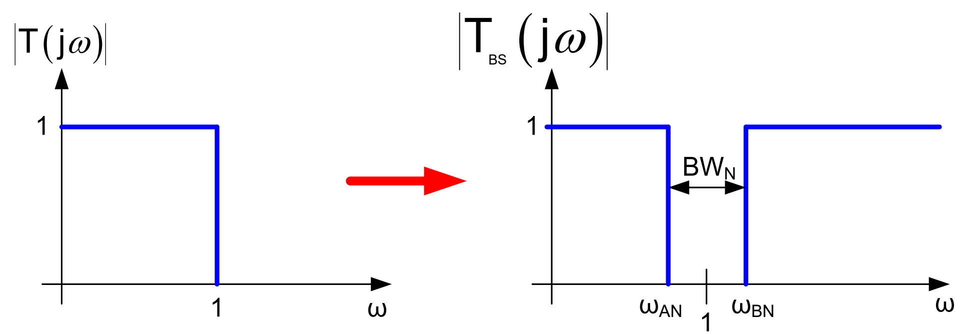
LP to BP

$$s \rightarrow \frac{s^2+1}{s \bullet BW_N}$$



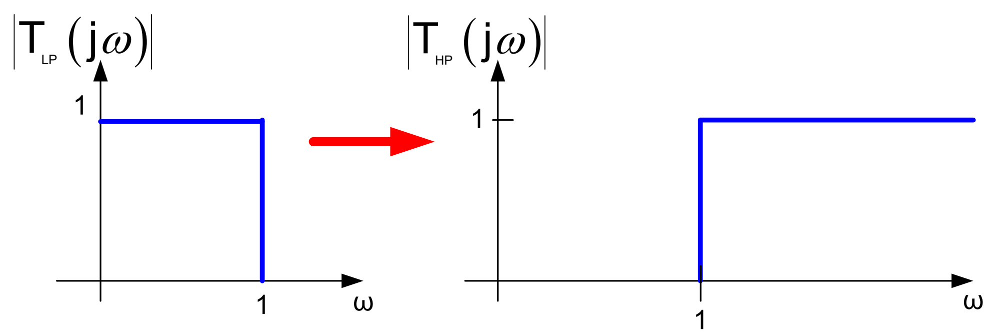
LP to BS

$$s \rightarrow \frac{s \bullet BW_N}{s^2+1}$$



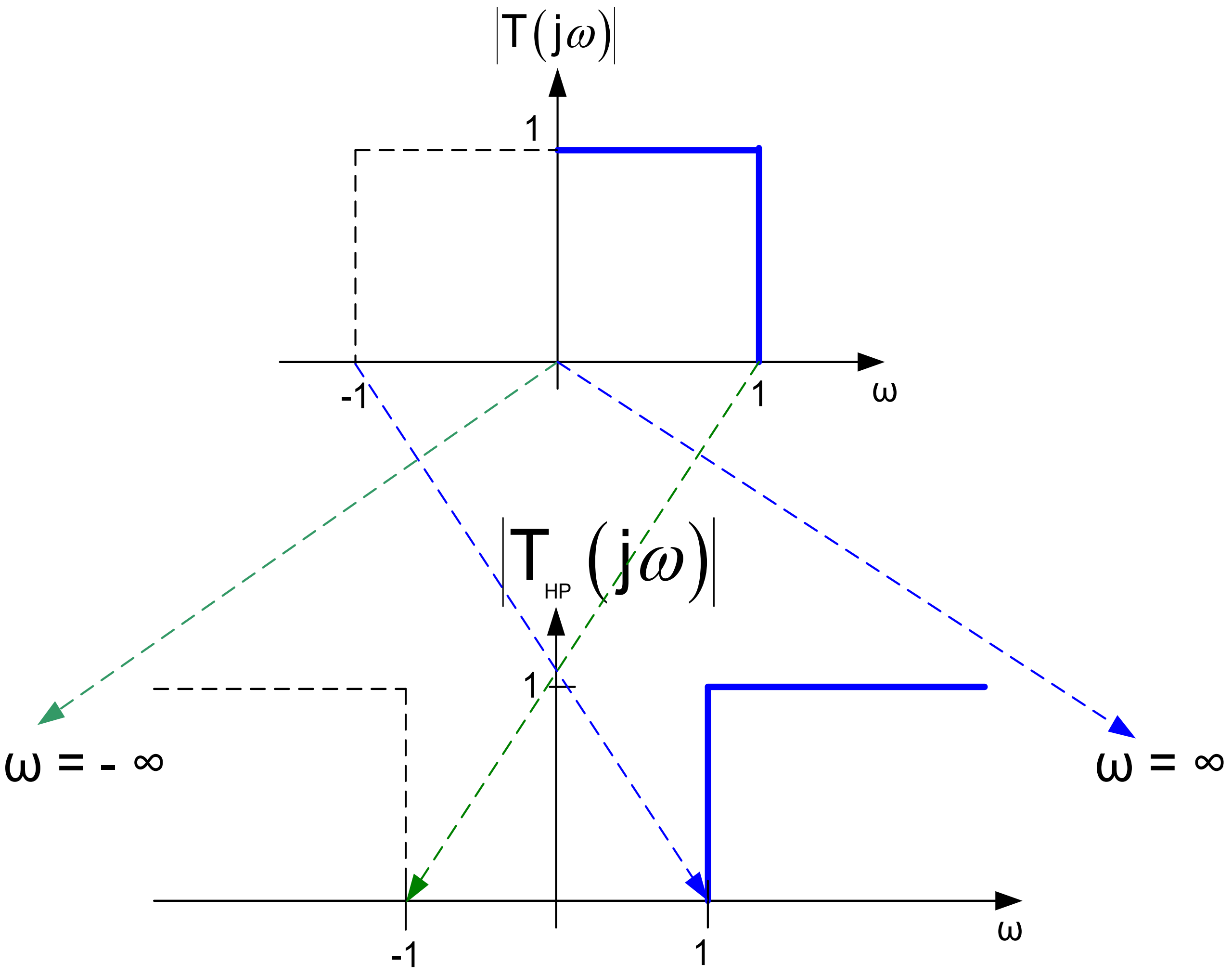
LP to HP

$$s \rightarrow \frac{1}{s}$$



LP to HP Transformation

(Normalized Transform)



Standard LP to HP Transformation

Frequency and s-domain Mappings

(subscript variable in LP approximation for notational convenience)

$T_{LPN}(s_x)$

$s_x \rightarrow \frac{1}{s}$

$T_{HPN}(s)$

$$s_x \rightarrow \frac{1}{s}$$

$$\omega_x \rightarrow \frac{-1}{\omega}$$

$$s \leftarrow \frac{1}{s_x}$$

$$\omega \leftarrow \frac{-1}{\omega_x}$$

Standard LP to HP Transformation

Pole Mappings

Claim: With a variable mapping transform, the variable mapping naturally defines the mapping of the poles of the transformed function

$$T_{LPN}(s_X)$$

$$s_X \downarrow \frac{1}{s}$$

$$T_{HPN}(s)$$

$$p_X \rightarrow \frac{1}{p}$$

$$p \leftarrow \frac{1}{p_X}$$

Standard LP to HP Transformation

Pole Mappings

$$T_{LPN}(s_x)$$

$$s_x \downarrow \frac{1}{s}$$

$$T_{HPN}(s)$$

$$p \leftarrow \frac{1}{p_x}$$

If $p_X=\alpha+j\beta$

$$p=\frac{1}{\alpha+j\beta}=\frac{\alpha-j\beta}{\alpha^2+\beta^2}$$

and $p_X=\alpha-j\beta$

$$p=\frac{1}{\alpha+j\beta}=\frac{\alpha+j\beta}{\alpha^2+\beta^2}$$

Standard LP to HP Transformation

Pole Mappings

$T_{LPN}(s_X)$

$s_X \rightarrow \frac{1}{s}$

$T_{HPN}(s)$

$$p \leftarrow \frac{1}{p_X}$$

If $p_X=\alpha+j\beta$ and $p_X=\alpha-j\beta$

$$p=\frac{1}{\alpha+j\beta}=\frac{\alpha-j\beta}{\alpha^2+\beta^2} \qquad p=\frac{1}{\alpha+j\beta}=\frac{\alpha+j\beta}{\alpha^2+\beta^2}$$

Highpass poles are scaled in magnitude but make identical angles with imaginary axis

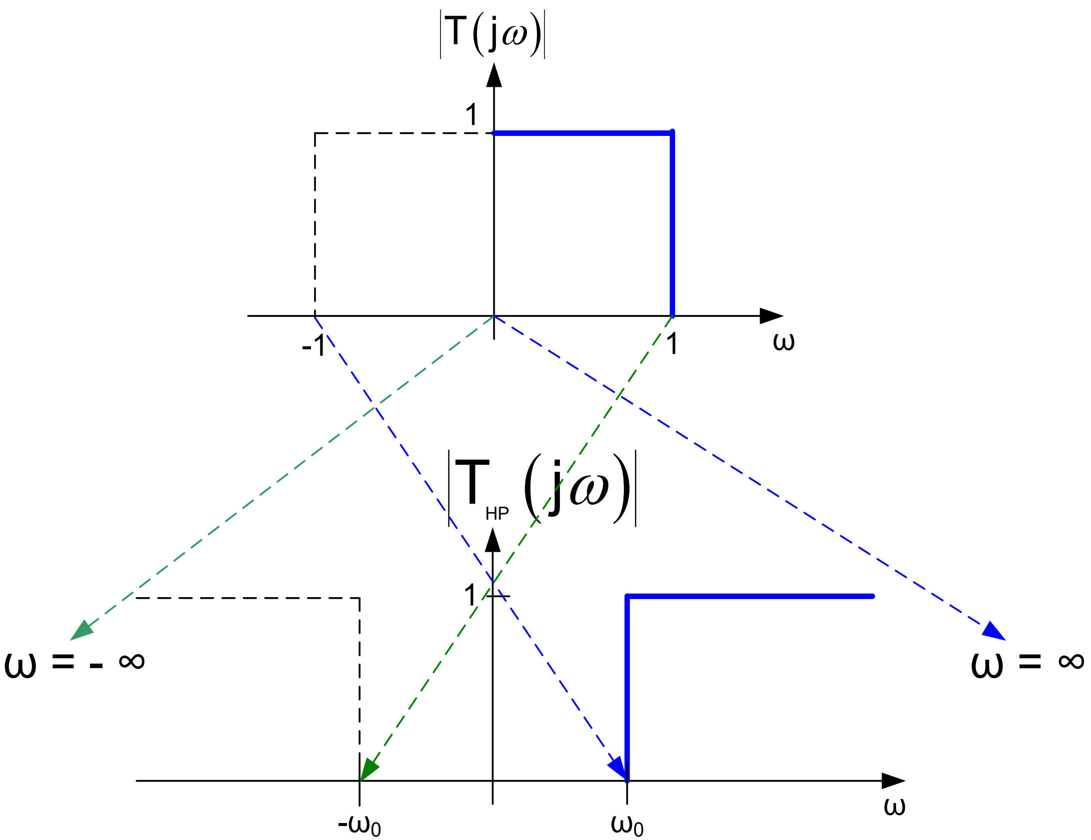
HP pole Q is same as LP pole Q

Order is preserved

Standard LP to HP Transformation

(Un-normalized variable mapping transform)

$$s \rightarrow \frac{\omega_0}{s}$$



Stay Safe and Stay Healthy !

End of Lecture 16